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Relativistic time of arrival and traversal time

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Abstract

The time of arrival and the traversal time through a region of a free particle with spin $\frac{1}{2}$ are computed by applying the relativistic extension of the event-enhanced quantum theory presented in a previous paper. There is a very good coincidence of the results of this formalism and the results obtained by using classical relativistic mechanics.

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1. Introduction

When does a particle arrive at a given point? This is a simple question but it cannot be answered unambiguously in the standard formulation of quantum mechanics. Nevertheless, there are many theoretical proposals to introduce a quantum time of arrival concept. A review can be, for example, found in Muga, Sala and Palao [1] and an extensive review including a lot of references in Muga and Leavens [2].

The related question of how long a particle needs to traverse a given finite region is also problematic in standard quantum mechanics. This time is called traversal time. It is often examined if a potential is in the given region and the particle must tunnel through the region. Reviews about 'traversal time' and 'tunnelling time' are given in [3–7].

It is possible to deal with these questions of 'time of arrival' and 'traversal time' by using the extension of standard quantum mechanics called event-enhanced quantum theory (EEQT) [8–11]. Its non-relativistic version was proposed by Blanchard and Jadczyk [12–14]. The main idea of EEQT is to view the total system as consisting of coupled classical and quantum parts. The pure states of the quantum part are wavefunctions which are not directly observable, whereas the pure states of the classical part can be observed without disturbing them. Changes of the classical pure states are discrete and irreversible, they are called events. A review about other applications of EEQT is, for example, given in [15].

In this paper, we shall examine the 'time of arrival' and the 'traversal time' using a relativistic extension of EEQT for one spin-$\frac{1}{2}$ particle. Let us consider a two-dimensional spacetime and a freely moving particle (except for the influence exerted by the detectors, see below). Blanchard and Jadczyk have introduced a relativistic extension of EEQT [16] using the idea of a proper time and an indefinite scalar product. In a previous paper [17], we have presented the alternative approach followed here.

In section 2, this approach will be summarized. Different possible initial states of the particle will be presented and discussed in section 3. In section 4, we shall compute the 'time of arrival', and the 'traversal time' in section 5. The paper will end with a conclusion.

2. A relativistic extension of EEQT

We shall first review the extension of EEQT proposed in [17]. It describes one spin-$\frac{1}{2}$ particle in a relativistic way and in four-dimensional spacetime. Here, let us restrict ourselves to a two-dimensional spacetime.

We postulate the existence of a supplementary, intrinsic time, called proper time τ. It is independent of the reference frame and plays the role of (absolute) time in non-relativistic quantum mechanics. As in EEQT, the total system consists of a classical and a quantum part which are coupled. At a given proper time τ, the (pure) state of the total system is a pair (ω_τ, Ψ_τ). ω_τ is the state of the classical part and Ψ_τ is the state of the quantum part.

A (pure) state of the classical part is a number $\omega_\tau \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and a change of the classical (pure) state will also be called 'event' as in non-relativistic EEQT.

The (pure) states of the quantum part will be (heuristically) solutions $\Psi : \mathbb{R} \times \mathbb{R} \to \mathbb{C}^4$ of the Dirac equation. Because only a free particle is examined in this paper, we use the free Dirac equation

$$\mathrm{i}\hbar c \frac{\partial}{\partial(ct)} \Psi(ct, x) = H_0 \Psi(ct, x) \tag{1}$$

with $H_0 = -\mathrm{i}c\hbar\gamma^0\gamma^1 \frac{\partial}{\partial x} + mc^2\gamma^0$ and the Dirac or standard representation of the γ-matrices (see e.g. [18]). The space of quantum states is denoted by $\mathcal{H}$. A more precise definition of $\mathcal{H}$ can be found in [17].

Now, let us introduce a positive-definite scalar product between two quantum states $\Psi_A, \Psi_B \in \mathcal{H}$:

$$\langle \Psi_A | \Psi_B \rangle_{\mathcal{H}} := \int_\sigma j^\mu_{AB} \, \mathrm{d} f_\mu \tag{2}$$

with σ an arbitrary space-like hyperplane, $j^\mu_{AB} = \Psi^+_A \gamma^0 \gamma^\mu \Psi_B$ and $\mathrm{d} f_\mu$ denotes the differential 'surface' element of σ. This scalar product is well defined because it is independent of σ. This follows from Gauss theorem and the fact that $\partial_\mu j^\mu_{AB} = 0$. Moreover, one can show that this scalar product is covariant, its value being independent of the reference frame.

We introduce the operators $U_{(ct_0,x_0)}$ with $ct_0, x_0 \in \mathbb{R}$:

$$(U_{(ct_0,x_0)}\Psi)(x) := \Psi(ct_0, x_0 + x).$$

An interesting property of a quantum state is that it is uniquely given by its values on a space-like hyperplane σ. Therefore, the operators $U_{(ct_0,x_0)}$ are invertible. $\Psi = U^{-1}_{(ct_0,x_0)}\psi$ is the solution of the free Dirac equation (1) fulfilling the initial condition $\Psi(ct_0, x) = \psi(x - x_0)$, so

$$\Psi(ct, x) = \left(U^{-1}_{(ct_0,x_0)}\psi\right)(ct, x) = \exp\left(-\frac{\mathrm{i}}{\hbar}(t - t_0)H_0\right)\psi(x - x_0).$$

Now, we shall formulate an algorithm for modelling continuous relativistic measurements, indeed detections of the particle, by rewriting the algorithm of EEQT, replacing t with τ and using the Hilbert space of 'solutions' $\mathcal{H}$.

The reference frame is denoted by K. Hereafter, the preparation event at proper time $\tau = \tau_0$ is assumed for simplicity to be associated with a spacetime point (ct_0, x_0). The initial particle state after the preparation should be Ψ_0. Let us consider n detectors with trajectories $z_j(\tau)$, $j = 1, \ldots, n$. The trajectories start at proper time $\tau = \tau_0$ from the backward light-cone of the spacetime point of the 'preparation event':

$$\left(ct_0 - z_j^0(\tau_0)\right)^2 - \left(x_0 - z_j^1(\tau_0)\right)^2 = 0 \qquad z_j^0(\tau_0) \leqslant ct_0.$$

Detections which happen in the past of the preparation time are possible, but only if the detection spacetime point is not located in the backward light-cone of the spacetime point of the preparation event. Similarly, we always demand that no event can be placed in the backward light-cone of the previous (concerning the proper time) event. Each detector is characterized by operators $G_j(\tau) : \mathcal{H} \to \mathcal{H}$. Let $G_j^+(\tau)$ be the adjoint operator. The total coupling between the quantum and the classical part is given by $\Lambda(\tau) := \sum_{j=1}^n G_j^+(\tau) G_j(\tau)$.

The detection algorithm is defined in the following way:

(i) The preparation event at proper time $\tau = \tau_0$ is associated with the spacetime point (ct_0, x_0). The quantum state is Ψ_{τ_0} with $\left\|\Psi_{\tau_0}\right\|_{\mathcal{H}}^2 \equiv \left\langle \Psi_{\tau_0} \middle| \Psi_{\tau_0} \right\rangle_{\mathcal{H}} = 1$ and the classical state is $\omega_{\tau_0} = 0$.

(ii) Choose uniformly a random number $r \in [0, 1]$.

(iii) Propagate the quantum state forward in proper time by solving

$$\frac{\partial}{\partial \tau} \Psi_\tau = -\frac{1}{2} \Lambda(\tau) \Psi_\tau \tag{3}$$

until $\tau = \tau_1$, where τ_1 is defined by

$$1 - \left\|\Psi_{\tau_1}\right\|_{\mathcal{H}}^2 = \int_{\tau_0}^{\tau_1} \mathrm{d}\tau \langle \Psi_\tau | \Lambda \Psi_\tau \rangle_{\mathcal{H}} = r.$$

Let $\omega_\tau = \omega_{\tau_0}$ until $\tau = \tau_1$, a detection happens at proper time $\tau = \tau_1$.

(iv) We choose the detector k—which detects the particle—with probability

$$p_k = \frac{1}{N} \left\|G_k(\tau_1) \Psi_{\tau_1}\right\|_{\mathcal{H}}^2$$

with $N = \sum_{j=1}^n \left\|G_j(\tau_1) \Psi_{\tau_1}\right\|_{\mathcal{H}}^2$.

(v) Let l be the detector which detects effectively the particle. The detection happens at the point $z_l(\tau_1)$. The detection induces the following change of the states:

$$\left(\omega_{\tau_1}, \Psi_{\tau_1}\right) \longrightarrow \left(l, \frac{G_l(\tau_1)\Psi_{\tau_1}}{\|G_l(\tau_1)\Psi_{\tau_1}\|_{\mathcal{H}}}\right).$$

The algorithm can start again perhaps with other detectors at position (ii).

Because the scalar product is covariant, this algorithm is covariant. Moreover, its non-relativistic limit reduces to the algorithm of the non-relativistic EEQT. If we 'charge conjugate' the initial state $\Psi_{\tau_0} \to \Psi_{\tau_0}^C \equiv C\gamma^{0^T} \Psi_{\tau_0}^*$ and the detector functions $G_j(\tau) \to C\gamma^{0^T} G_j^*(\tau) \gamma^{0^T} C^+$ with $C = \mathrm{i}\gamma^2\gamma^0$, then the algorithm will give the same detections as if we start with Ψ_{τ_0} and $G_j(\tau)$ (choosing the same random numbers). The quantum state in the 'charge conjugated' world Ψ_τ^C and the quantum state in the 'normal' world are always connected by $\Psi_\tau^C = C\gamma^{0^T} \Psi_\tau^*$.

Note, that also an algorithm for modelling ideal measurements of infinitesimal small duration is formulated in [17]. It can be seen as playing the role of a relativistic, covariant reduction postulate.

3. Initial quantum state

We examine three different initial states of the particle in this paper. Remember that an initial state of the particle must be a solution of the Dirac equation (1).

The first state corresponds to a state with only positive energies:

$$\Psi_{0,P}(ct,x) = \frac{1}{N_P}\int \mathrm{d}k \frac{1}{2\hat{E}} F_{\Delta k}\left(k - \frac{p_0}{\hbar}\right)\begin{pmatrix}\hat{E}+\hat{m}\\ 0\\ 0\\ k\end{pmatrix}\exp(\mathrm{i}k(x-x_0) - \mathrm{i}\hat{E}ct)$$

with $\hat{m} = \frac{mc}{\hbar}$, $\hat{E} = \sqrt{k^2+\hat{m}^2}$,

$$F_{\Delta k}(k) = \begin{cases}\exp\left(-\frac{k^2}{\Delta k^2 - k^2}\right) & \text{for} \quad |k| < \Delta k\\ 0 & \text{otherwise}\end{cases}$$

and N_P being a normalization factor so that $\|\Psi_{0,P}\|^2_{\mathcal{H}} = 1$. This state describes an electron with charge $-e$.

The second one corresponds to a state with only negative energies:

$$\Psi_{0,N}(ct,x) = \frac{1}{N_N}\int \mathrm{d}k \frac{1}{2\hat{E}} F_{\Delta k}\left(k - \frac{p_0}{\hbar}\right)\begin{pmatrix}\hat{E}-\hat{m}\\ 0\\ 0\\ k\end{pmatrix}\exp(-\mathrm{i}k(x-x_0) + \mathrm{i}\hat{E}ct)$$

with N_N being a normalization factor so that $\|\Psi_{0,N}\|^2_{\mathcal{H}} = 1$. Note that the above algorithm is invariant under charge conjugation. Considering the charge conjugate of the initial state and the detector functions, we get the same events. Because the 'charge conjugated' world and the 'normal world' should describe the same physical situation and because the charge conjugation of $\Psi_{0,N}$ describes a particle with charge $+e$ in the 'charge conjugated world', we demand that the initial state $\Psi_{0,N}$ describes a positron with charge $+e$ also in the 'normal' world.

As the third initial state, let us consider a mixed state:

$$\begin{aligned}\Psi_{0,PN}(ct,x) &= U^{-1}_{(0,x_0)}\left[\frac{1}{(2\pi)^{1/4}\sqrt{\eta}}\exp\left(-\frac{x^2}{4\eta^2} + \mathrm{i}\frac{p_0}{\hbar}x\right)\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right](ct,x)\\ &= \frac{\sqrt{2\eta}}{(2\pi)^{3/4}}\int \mathrm{d}k \frac{1}{2\hat{E}}\exp\left(-\eta^2\left(k-\frac{p_0}{\hbar}\right)^2\right)\begin{pmatrix}\hat{E}+\hat{m}\\ 0\\ 0\\ k\end{pmatrix}\exp(\mathrm{i}k(x-x_0) - \mathrm{i}\hat{E}ct)\\ &\quad + \frac{\sqrt{2\eta}}{(2\pi)^{3/4}}\int \mathrm{d}k \frac{1}{2\hat{E}}\exp\left(-\eta^2\left(k+\frac{p_0}{\hbar}\right)^2\right)\begin{pmatrix}\hat{E}-\hat{m}\\ 0\\ 0\\ k\end{pmatrix}\\ &\quad \times \exp(-\mathrm{i}k(x-x_0) + \mathrm{i}\hat{E}ct).\end{aligned}$$

The constants are fixed in such a way that $\|\Psi_{0,PN}\|^2_{\mathcal{H}} = 1$. We will choose $\eta \gg \frac{\hbar}{2mc}$ with $\frac{\hbar}{2mc} \approx 0.002\,\text{Å}$ being the approximate amplitude of a zitterbewegung (see for example [18]). Therefore, zitterbewegung will not be visible. Nevertheless, there must be a physical interpretation of the mixed state: we assume that the particle (a single particle) can be in an 'electron-state' (solution with positive energies) and in a 'positron-state' (solution with negative energies), in analogy with the case that a particle can be, e.g. in a spin $+\frac{1}{2}$-state or in a spin $-\frac{1}{2}$-state. Superpositions as $\Psi_{0,PN}$ of the two states should be (in analogy with the spin-case) possible and allowed.

4. Free time of arrival

In this section, the above algorithm is applied to simulate the detection of the particle by one detector at rest. Let us use the reference frame K_0 in which the detector is at rest. The particle should move freely (except for the influence exerted on it by the detector) in positive x direction. The preparation event at proper time $\tau_0 = 0$ is associated with a spacetime point $(0, x_0)$. The mean momentum of the particle is p_0. The detector is put at x_D with $x_D > x_0$. Its trajectory is $z(\tau) = (c\tau + x_0 - x_D, x_D)$. It measures the time of arrival of the particle at x_D. The coupling operator should be given by

$$G(\tau) = U_{z(\tau)}^{-1} g(x) U_{z(\tau)}$$

with $g(x)$ characterizing the sensitivity of the detector:

$$g(x) = \sqrt{\frac{2W_D}{\hbar}} F_{\frac{\Delta x_D}{2}}(x).$$

The adjoint operator is $G^{+}(\tau) = U_{z(\tau)}^{-1} g^{+}(x) U_{z(\tau)}$. Because it is possible that the particle is never detected, we stop the algorithm at $\tau = \tau_{\mathrm{CUT}}$ (with τ_{CUT} large).

Since the algorithm is covariant, the choice of K_0 as the reference frame is arbitrary. The algorithm can be applied in any reference frame, and there will result (if we choose the same random numbers) the same events in all reference frames.

Using our algorithm, the probability that the detector detects the particle at all is

$$P_\infty = \int_0^{\tau_{\mathrm{CUT}}} \mathrm{d}\tau \, \langle \Psi_\tau | \Lambda \Psi_\tau \rangle_{\mathcal{H}}.$$

The probability density for a 'proper time of arrival' at the detector is given by ($\tau < \tau_{\mathrm{CUT}}$)

$$p(\tau) = \frac{1}{P_\infty} \langle \Psi_\tau | \Lambda \Psi_\tau \rangle_{\mathcal{H}}.$$

It is zero for $\tau \leqslant 0$ and $\tau \geqslant \tau_{\mathrm{CUT}}$. Using this probability density for 'proper time of arrival', the probability density and the expectation value for the time of arrival can be calculated in an arbitrary reference frame.

Let us first look at the detector's rest frame K_0. If a detection happens at proper time τ, then it happens in spacetime point $z(\tau) = (c\tau + x_0 - x_D, x_D)$. This implies a time of arrival of $t = \tau - \frac{x_D - x_0}{c}$. So we get the following probability density for the time of arrival in the detector's rest frame K_0:

$$\varrho_0(t) = p\left(t + \frac{x_D - x_0}{c}\right).$$

The expectation value (or mean time of arrival) is

$$T_{a,0} = \int \mathrm{d}t \, t \varrho_0(t) = \int \mathrm{d}\tau \left(\tau - \frac{x_D - x_0}{c}\right) p(\tau) = \int \mathrm{d}\tau \, \tau p(\tau) - \frac{x_D - x_0}{c}.$$

Now, we want to calculate these values in a reference frame K_v which moves with velocity v with respect to the detector's rest frame K_0. The Poincaré-transformation $K_0 \to K_v$ has the following form:

$$\tilde{x} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \begin{pmatrix} 1 & -\frac{v}{c} \\ -\frac{v}{c} & 1 \end{pmatrix} x.$$

The detector trajectory in K_v is

$$\tilde{z}(\tau) = \left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} \left(c\tau + x_0 - x_D - \frac{v}{c}x_D, \quad -v\tau - \frac{v}{c}x_0 + \frac{v}{c}x_D + x_D\right).$$

So the normalized probability density for the time of arrival in the reference frame K_v is given by

$$\varrho_v(\tilde{t}) = \sqrt{1-\frac{v^2}{c^2}}\, p\left(\sqrt{1-\frac{v^2}{c^2}}\tilde{t} + \frac{x_D - x_0}{c} + \frac{v}{c^2}x_D\right)$$

and the expectation value (or mean time of arrival) in K_v is

$$T_{a,v} = \int \mathrm{d}\tilde{t}\, \tilde{t} \varrho_v(\tilde{t}) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left[T_{a,0} - \frac{v}{c^2}x_D\right]. \tag{4}$$

4.1. Numerical approach

The reference frame K_0 is used to compute $p(\tau)$. Therefore, we define

$$\Omega(\tau, x) := (U_{z(\tau)}\Psi_\tau)(x) = \Psi_\tau(c\tau + x_0 - x_D, x_D + x).$$

If Ψ_τ is a solution of (1) and (3), then

$$\begin{aligned} \mathrm{i}\hbar\frac{\partial}{\partial\tau}\Omega(\tau, x) &= \mathrm{i}\hbar c \frac{\partial}{\partial(ct)}\Psi_\tau(\overbrace{c\tau + x_0 - x_D}^{ct}, x_D + x) + \mathrm{i}\hbar\left(\frac{\partial\Psi_\tau}{\partial\tau}\right)(c\tau + x_0 - x_D, x_D + x) \\ &= H_0\Omega(\tau, x) - \mathrm{i}\frac{\hbar}{2}g^+(x)g(x)\Omega(\tau, x). \end{aligned} \tag{5}$$

This equation must be solved with the initial condition $\Omega(0, x) = \Psi_0(x_0 - x_D, x_D + x)$. Then, P_∞ and $p(\tau)$ can be calculated because $\langle\Psi_\tau|\Lambda\Psi_\tau\rangle_{\mathcal{H}} = \int \mathrm{d}x\, \Omega^+(\tau, x)g^+(x)g(x)\Omega(\tau, x)$. Using $p(\tau)$, we get $\varrho_0(t)$ and $T_{a,0}$.

The proper time dynamics of Ω is approximated by

$$\Omega(\tau + \Delta\tau) \approx \exp\left(-\frac{\Delta\tau}{2}\frac{1}{2}g^+(x)g(x)\right)\exp\left(-\Delta\tau\frac{\mathrm{i}}{\hbar}H_0\right)\exp\left(-\frac{\Delta\tau}{2}\frac{1}{2}g^+(x)g(x)\right)\Omega(\tau).$$

We discretize the proper time and space with steps $\Delta x_B = c\Delta\tau_B = 0.0004$ Å. Then, the first and the last operators can be computed directly. The second operator is discretized by using the method of Wessels, Caspers and Wiegel [19]. The boundary conditions are walls at $x = -6$ Å and at $x = 4$ Å in such a way that $\Omega(\tau, -6\,\text{Å}) = \Omega(\tau, 4\,\text{Å}) = 0$ for all τ. Let $\tau_{\mathrm{CUT}} = 13.0\,\text{Å}/c\,(p_0 < 0.5\,mc)$, $\tau_{\mathrm{CUT}} = 7.0\,\text{Å}/c\,(0.5\,mc \leqslant p_0 < 0.75\,mc)$, $\tau_{\mathrm{CUT}} = 5.0\,\text{Å}/c\,(0.75\,mc \leqslant p_0 < 1.0\,mc)$, $\tau_{\mathrm{CUT}} = 4.5\,\text{Å}/c\,(1.0\,mc \leqslant p_0)$. The simulations is done again with other time and space steps $\Delta x_A = c\Delta\tau_A = 0.0006$ Å. So the error in the expectation value $T_{a,0}$ can be approximated by

$$\mathrm{error}(T_{a,0}) = \pm\frac{\Delta x_B}{\Delta x_A - \Delta x_B}|T_{a,0}(\Delta x_B) - T_{a,0}(\Delta x_A)|. \tag{6}$$

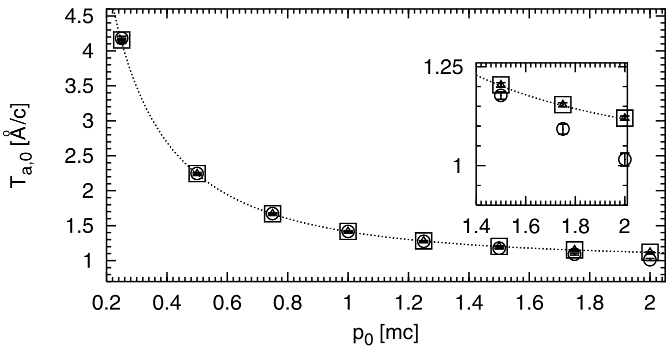

Figure 1. Mean time of arrival $T_{a,0}$ versus particle momentum p_0 in the detector's rest frame K_0, relativistic simulation with detector parameters $\Delta x_D = 0.01$ Å, $W_D = 1 \times 10^{-5}\, mc^2$ started with different initial states : $\Psi_{0,P}$ (boxes with error bars), $\Psi_{0,N}$ (triangles with error bars), $\Psi_{0,PN}$ (circles with error bars), other parameters see text; classical relativistic mechanics $t_{a,RM}$ (dotted line); the figure inside is a zoom of the right lower area of the figure outside.

4.2. Results

We set $x_0 = -1$ Å, $\Delta k = 10\,\text{Å}^{-1}$, $\eta = 0.1$ Å, $x_D = 0$ Å, $\Delta x_D = 0.01$ Å and $W_D = 1 \times 10^{-5}\, mc^2$.

Figure 1 shows the corresponding expectation values of the time of arrival $T_{a,0}$ in the detector's rest frame K_0 for different momenta p_0 and for the three different initial states. The error bars are calculated using (6). In addition, figure 1 shows the arrival times calculated by using the classical relativistic mechanics of a point-particle:

$$t_{a,RM} = \frac{x_D - x_0}{c}\sqrt{1 + \frac{m^2c^2}{p_0^2}}.$$

The expectation values are nearly independent of the initial state $\Psi_{0,PN}$, $\Psi_{0,P}$ or $\Psi_{0,N}$. Furthermore, there is good agreement between the values we computed and the results obtained by using classical relativistic mechanics. Only for very high momenta, the expectation values of the simulation with $\Psi_{0,PN}$ are a bit smaller than the times from classical mechanics and those obtained by the simulations with other initial states.

The reason can be seen in figure 2, which shows probability densities in the detector's rest frame K_0. For $p_0 = 2.0\, mc$ and $\Psi_{0,PN}$, there is a small probability for negative times of arrival due to the negative momentum components of the initial state $\Psi_{0,PN}$. The cut at $t = -1$ Å$/c$ results from the restriction that an event cannot be placed in the backward light-cone of the previous event (see section 2). The small probabilities for negative times of arrival explain why the expectation values of the simulation with $\Psi_{0,PN}$ are smaller than the results of classical mechanics and those of the other simulations. We also see that the probability densities are (nearly) the same if we start with $\Psi_{0,P}$ or $\Psi_{0,N}$.

The expectation values $T_{a,v}$ in different reference frames are connected by (4). Note that in classical relativistic mechanics the time of arrival $\tilde{t}_{a,RM}$ in the reference frame K_v is connected to the result $t_{a,RM}$ in the reference frame K_0 in the same manner (compare with (4)):

$$\tilde{t}_{a,RM} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\left[t_{a,RM} - \frac{v}{c^2}x_D\right].$$

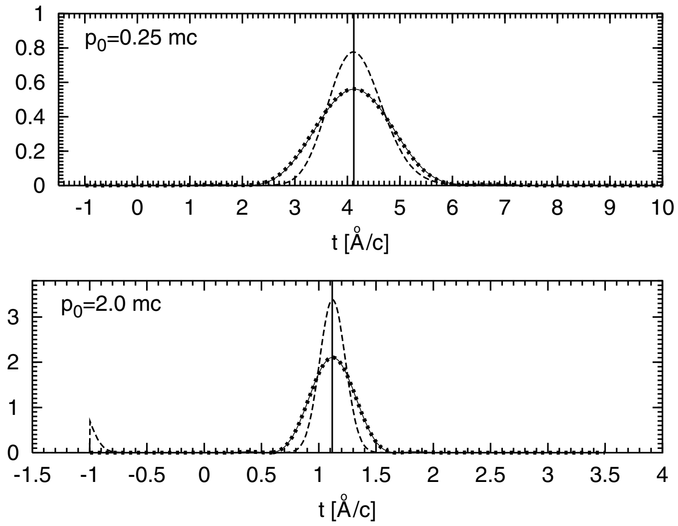

Figure 2. Probability densities ϱ_0 for the time of arrival in the detector's rest frame K_0, detector parameters: $\Delta x_D = 0.01$ Å, $W_D = 1 \times 10^{-5}\, mc^2$, initial state: $\Psi_{0,P}$ (small solid line), $\Psi_{0,N}$ (big dotted line), $\Psi_{0,PN}$ (dashed line), particle momentum p_0; the vertical solid line indicates the arrival time deduced from classical relativistic mechanics.

Another important and interesting question is how the expectation values depend on the parameters of the detector. The initial state is now the function $\Psi_{0,P}$ with positive energies. We shall examine those particle momenta which are also examined in figure 1 and compute the probability densities and the expectation values for four different pairs of detector parameters.

First, using a 'higher' detector ($\Delta x_D = 0.01$ Å, $W_D = 1.0\, mc^2$), the expectation values and the normalized probability densities do not change for all examined particle momenta. The detection probability P_∞ increases with increasing detector height W_D.

Next, using a wider detector ($\Delta x_D = 0.4$ Å, $W_D = 1 \times 10^{-5}\, mc^2$), the expectation values again do not change. The normalized probability density does not change in a significant way, it only becomes a bit wider. Again the detection probability P_∞ increases with increasing detector width Δx_D.

The results only change for a very wide and height detector ($\Delta x_D = 0.4$ Å, $W_D = 1.0\, mc^2$). The expectation values and the normalized probability densities are then shifted to earlier times.

In summary, the simulations show a wide range of detector parameters for which the results do not change significantly.

5. Free traversal time

Using two detectors at rest one behind the other, it is possible to measure the traversal time through the region located between the two detectors. Let us use the detectors' rest frame K_0. The preparation event at proper time $\tau = 0$ is associated with a spacetime point $(0, x_0)$. The particle moves in positive x direction. We put a detector D_1 at x_1 with $x_1 > x_0$. Its trajectory is $z_1(\tau) = (c\tau + x_0 - x_1, x_1)$. This detector can detect the particle without destroying it. A second detector D_2 is put at x_2 with $x_2 > x_1$. Its trajectory is $z_2(\tau) = (c\tau + x_0 - x_2, x_2)$. It destroys the particle after detection.

At the beginning of the measurement, both detectors D_1 and D_2 are active. If detector D_1 detects the particle, it turns itself off, but detector D_2 stays on. If detector D_2 detects the particle, the experiment is completed. Thus, the particle can be detected by detector D_1 at a time t_1 and then by detector D_2 at a time t_2. If this happens, the time difference $t_2 - t_1$ is defined to be the 'traversal time'.

It is also possible that the particle is detected by D_2 without prior detection by D_1, but this situation should not contribute to traversal times. Moreover, it is possible that the particle is never detected or only detected one time by detector D_1. For this reason the experiment or simulation should be stopped after a reasonable and finite period of time τ_{CUT} (with τ_{CUT} large).

This measurement is simulated by applying the algorithm described in section 2. The coupling operators of detector D_j should be given by

$$G_j(\tau) = U^{-1}_{z_j(\tau)} g_j(x) U_{z_j(\tau)} \qquad j = 1, 2$$

with $g(x)$ characterizing the sensitivity of the detector D_j:

$$g_j(x) = \sqrt{\frac{2W_j}{\hbar}} F_{\frac{\Delta x_j}{2}}(x).$$

Let Ψ_0 be the initial state and Ψ_τ the solution of (1) and (3). Then, the probability that the particle is detected by D_1 at all is

$$P_{\infty,1} = \int_0^{\tau_{\mathrm{CUT}}} \mathrm{d}\tau \big\langle \Psi_\tau \big| G_1^+(\tau) G_1(\tau) \Psi_\tau \big\rangle_{\mathcal{H}}.$$

The probability density that the particle is detected by D_1 is given by ($\tau < \tau_{\mathrm{CUT}}$)

$$p_1(\tau) = \frac{1}{P_{\infty,1}} \big\langle \Psi_\tau \big| G_1^+(\tau) G_1(\tau) \Psi_\tau \big\rangle_{\mathcal{H}}.$$

If a detection by detector D_1 happens at τ_1, the quantum state after the detection is given by

$$\Phi^{(\tau_1)}_{\tau_1} := \frac{G_1(\tau_1)\Psi_{\tau_1}}{\|G_1(\tau_1)\Psi_{\tau_1}\|_{\mathcal{H}}}. \tag{7}$$

Let $\Phi^{(\tau_1)}_\tau$ be the solution of (3) with initial state (7). The conditional probability that the particle is detected a second time by D_2 if it is detected by D_1 at τ_1 is

$$P^{(\tau_1)}_\infty = \int_{\tau_1}^{\tau_{\mathrm{CUT}}} \mathrm{d}\tau_2 \big\langle \Phi^{(\tau_1)}_{\tau_2} \big| G_2^+(\tau_2) G_2(\tau_2) \Phi^{(\tau_1)}_{\tau_2} \big\rangle_{\mathcal{H}}$$

and the probability density for a second detection at proper time τ_2 by detector D_2 after a detection of detector D_1 at proper time τ_1 is given by

$$p_2^{(\tau_1)}(\tau_2) = \frac{1}{P^{(\tau_1)}_\infty} \big\langle \Phi^{(\tau_1)}_{\tau_2} \big| G_2^+(\tau_2) G_2(\tau_2) \Phi^{(\tau_1)}_{\tau_2} \big\rangle_{\mathcal{H}}.$$

Finally, the probability density for a first detection by D_1 at τ_1 and a second detection by D_2 at τ_2 is

$$\begin{aligned}
p_{12}(\tau_1, \tau_2) &= \frac{p_2^{(\tau_1)}(\tau_2) P^{(\tau_1)}_\infty p_1(\tau_1) P_{\infty,1}}{\int_0^{\tau_{\mathrm{CUT}}} \mathrm{d}\tau\, P^{(\tau)}_\infty p_1(\tau) P_{\infty,1}} \\
&= \frac{1}{P_{\infty,12}} \begin{cases} \big\langle \Phi^{(\tau_1)}_{\tau_2} \big| G_2^+(\tau_2) G_2(\tau_2) \Phi^{(\tau_1)}_{\tau_2} \big\rangle_{\mathcal{H}} \big\langle \Psi_{\tau_1} \big| G_1^+(\tau_1) G_1(\tau_1) \Psi_{\tau_1} \big\rangle_{\mathcal{H}} \\ \qquad \text{for } 0 < \tau_1 < \tau_{\mathrm{CUT}} \text{ and } \tau_1 < \tau_2 < \tau_{\mathrm{CUT}} \\ 0 \qquad \text{otherwise} \end{cases}
\end{aligned}$$

with $P_{\infty,12}$ being the probability that the particle is detected two times:

$$P_{\infty,12} = P_{\infty,1} \int_0^{\tau_{\mathrm{CUT}}} \mathrm{d}\tau\, P_\infty^{(\tau)} p_1(\tau)$$
$$= \int_0^{\tau_{\mathrm{CUT}}} \mathrm{d}\tau_1 \int_{\tau_1}^{\tau_{\mathrm{CUT}}} \mathrm{d}\tau_2 \left\langle \Phi_{\tau_2}^{(\tau_1)} \middle| G_2^+(\tau_2) G_2(\tau_2) \Phi_{\tau_2}^{(\tau_1)} \right\rangle_{\mathcal{H}} \left\langle \Psi_{\tau_1} \middle| G_1^+(\tau_1) G_1(\tau_1) \Psi_{\tau_1} \right\rangle_{\mathcal{H}}.$$

Note, that this probability density is independent of the reference frame in which the algorithm is applied.

We now calculate traversal times in different reference frames. In contrast to p_{12}, the probability density for traversal time depends on the reference frame.

Let us start with the detectors' rest frame K_0. If the first detection of D_1 happens at proper time τ_1, then it happens at spacetime point $z_1(\tau_1) = (c\tau_1 + x_0 - x_1, x_1)$. If the second detection of D_2 happens at proper time τ_2, then it happens at spacetime $z_2(\tau_2) = (c\tau_2 + x_0 - x_2, x_2)$. The resulting traversal time is, therefore,

$$t = \tau_2 + \frac{x_0 - x_2}{c} - \tau_1 - \frac{x_0 - x_1}{c} = \tau_2 - \tau_1 - \frac{x_2 - x_1}{c}.$$

So the normalized probability density for the traversal time in the detectors' rest frame K_0 is given by

$$\rho_0(t) = \int \mathrm{d}\tau\, p_{12}\left(\tau, t + \frac{x_2 - x_1}{c} + \tau\right).$$

The expectation value of the traversal time (or mean traversal time) in K_0 is

$$T_{t,0} = \int \mathrm{d}t\, t \int \mathrm{d}\tau\, p_{12}\left(\tau, t + \frac{x_2 - x_1}{c} + \tau\right).$$

Next, these values are calculated in the reference frame K_v (the reference frame which moves with velocity v with respect to the detectors' rest frame K_0). The detector trajectories in K_v are

$$\tilde{z}_1(\tau) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \left(c\tau + x_0 - x_1 - \frac{v}{c}x_1, -v\tau - \frac{v}{c}(x_0 - x_1) + x_1\right)$$
$$\tilde{z}_2(\tau) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \left(c\tau + x_0 - x_2 - \frac{v}{c}x_2, -v\tau - \frac{v}{c}(x_0 - x_2) + x_2\right).$$

If the first detection of D_1 happens at τ_1 and the second detection of D_2 happens at τ_2, then it results in a traversal time of

$$\tilde{t} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \left(\tau_2 - \tau_1 - \frac{x_2 - x_1}{c} - \frac{v}{c^2}(x_2 - x_1)\right).$$

So the normalized probability density for the traversal time in the reference frame K_v is given by

$$\rho_v(\tilde{t}) = \sqrt{1 - \frac{v^2}{c^2}} \int \mathrm{d}\tau\, p_{12}\left(\tau, \sqrt{1 - \frac{v^2}{c^2}}\tilde{t} + \frac{x_2 - x_1}{c} + \frac{v}{c^2}(x_2 - x_1) + \tau\right).$$

The expectation value of the traversal time (or mean traversal time) in K_v is

$$T_{t,v} = \int \mathrm{d}t\, t \sqrt{1 - \frac{v^2}{c^2}} \int \mathrm{d}\tau\, p_{12}\left(\tau, \sqrt{1 - \frac{v^2}{c^2}}\tilde{t} + \frac{x_2 - x_1}{c} + \frac{v}{c^2}(x_2 - x_1) + \tau\right)$$
$$= \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \left[T_{t,0} - \frac{v}{c^2}(x_2 - x_1)\right]. \tag{8}$$

5.1. Numerical approach

Let us use the reference frame K_0. For computation of the algorithm until the first detection, we define

$$\Omega_A(\tau, x) := (U_{(c\tau+x_0-x_1,0)}\Psi_\tau)(x) = \Psi_\tau(c\tau + x_0 - x_1, x)$$

Ψ_τ should be a solution of (1) and (3), so

$$\mathrm{i}\hbar\frac{\partial}{\partial\tau}\Omega_A(\tau, x) = H_0\Omega_A - \mathrm{i}\frac{\hbar}{2}g_1^+(x - x_1)g_1(x - x_1)\Omega_A - \mathrm{i}\frac{\hbar}{2}Tg_2(x - x_2)^+g_2(x - x_2)T^{-1}\Omega_A \tag{9}$$

with $T = U_{(c\tau+x_0-x_1,0)}U^{-1}_{(c\tau+x_0-x_2,0)} = \exp(-(x_2 - x_1)\frac{\mathrm{i}}{c\hbar}H_0)$. A solution of this equation has to be found satisfying the initial condition $\Omega_A(0, x) = \Psi_0(x_0 - x_1, x)$. Equation (9) is solved numerically with the proper time dynamics approximated by

$$\begin{aligned}\Omega_A(\tau + \Delta\tau) \approx \exp&\left(-\frac{\Delta\tau}{2}\frac{\mathrm{i}}{\hbar}mc^2\gamma^0 - \frac{\Delta\tau}{2}\frac{1}{2}g_1^+(x - x_1)g_1(x - x_1)\right)\exp\left(-\frac{\Delta\tau}{2}\frac{\mathrm{i}}{\hbar}\right.\\ &\times\left.\left(-\mathrm{i}\hbar c\gamma^0\gamma^1\frac{\partial}{\partial x}\right)\right)T\exp\left(-\Delta\tau\frac{1}{2}g_2^+(x - x_2)g_2(x - x_2)\right)T^{-1}\\ &\times\exp\left(-\frac{\Delta\tau}{2}\frac{\mathrm{i}}{\hbar}\left(-\mathrm{i}\hbar c\gamma^0\gamma^1\frac{\partial}{\partial x}\right)\right)\exp\left(-\frac{\Delta\tau}{2}\frac{\mathrm{i}}{\hbar}mc^2\gamma^0\right.\\ &\left.-\frac{\Delta\tau}{2}\frac{1}{2}g_1^+(x - x_1)g_1(x - x_1)\right)\Omega_A(\tau)\end{aligned}$$

with $T \approx \prod\exp\left(-\Delta\tau\frac{\mathrm{i}}{\hbar}H_0\right)$. We discretize the proper time and space with steps $\Delta\tau$ and Δx ($c\Delta\tau = \Delta x$). The boundary conditions are walls at $x = -8$ Å and $x = 8$ Å in such a way that $\Omega_A(\tau, -8\,\text{Å}) = \Omega_A(\tau, 8\,\text{Å}) = 0$ for all τ. All operators (including T) can be evaluated directly or are approximated by using the method of Wessels, Caspers and Wiegel [19] or by using Wendroff's formula (see e.g. [20]).

For simulating the second part of the algorithm (after a first detection by detector D_1 at proper time τ_1), we define

$$\Omega_B^{(\tau_1)}(\tau, x) := (U_{(c\tau+x_0-x_2,0)}\Psi_\tau)(x) = \Psi_\tau(c\tau + x_0 - x_2, x)$$

with Ψ_τ being a solution of (1) and (3) and get

$$\mathrm{i}\hbar\frac{\partial}{\partial\tau}\Omega_B^{(\tau_1)}(\tau, x) = H_0\Omega_B^{(\tau_1)}(\tau, x) - \mathrm{i}\frac{\hbar}{2}g_2^+(x - x_2)g_2(x - x_2)\Omega_B^{(\tau_1)}(\tau, x). \tag{10}$$

This equation must be solved with the initial condition

$$\Omega_B^{(\tau_1)}(\tau_1, x) = \frac{T^{-1}g_1(x - x_1)\Omega_A(\tau_1, x)}{\sqrt{\int \mathrm{d}x\,\Omega_A^+(\tau_1, x)g_1^+(x - x_1)g_1(x - x_1)\Omega_A(\tau_1, x)}}. \tag{11}$$

This can be done approximately in analogy with section 4.1. Using $\Omega_A(\tau, x)$ and $\Omega_B^{(\tau_1)}(\tau, x)$, we can calculate $P_{\infty,12}$, $p_{12}(\tau_1, \tau_2)$, $\rho_0(t)$ and $T_{t,0}$.

The computation is carried out with proper time and space step $c\Delta\tau_B = \Delta x_B = 0.0006$. The value of τ_{CUT} depends on the particle momentum: $\tau_{\mathrm{CUT}} = 31.5$ Å$/c$ ($p_0 = 0.25\,mc$), $\tau_{\mathrm{CUT}} = 17.5$ Å$/c$ ($p_0 = 0.5\,mc$), $\tau_{\mathrm{CUT}} = 13.5$ Å$/c$ ($p_0 = 0.75\,mc$), $\tau_{\mathrm{CUT}} = 11.5$ Å$/c$ ($1.0\,mc \leqslant p_0 < 1.5\,mc$), $\tau_{\mathrm{CUT}} = 10.5$ Å$/c$ ($1.5\,mc \leqslant p_0$). Moreover, the computation is done with proper time and space step $c\Delta\tau_A = \Delta x_A = 0.001$. So the error in the expectation value $T_{t,0}$ can be approximated by

$$\mathrm{error}(T_{t,0}) = \pm\frac{\Delta x_B}{\Delta x_A - \Delta x_B}|T_{t,0}(\Delta x_B) - T_{t,0}(\Delta x_A)|. \tag{12}$$

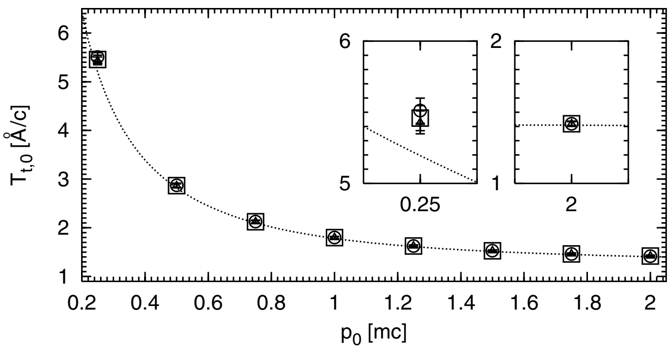

Figure 3. Mean traversal time $T_{t,0}$ versus particle momentum p_0 in the detectors' rest frame K_0, starting with different initial states : $\Psi_{0,P}$ (boxes with error bars), $\Psi_{0,N}$ (triangles with error bars), $\Psi_{0,PN}$ (circles with error bars), other parameters see text; results from classical relativistic mechanics $t_{t,RM}$ (dotted line).

The error in the probability $P_{\infty,12}$ is approximated by a similar formula:

$$\text{error}(P_{\infty,12}) = \pm \frac{\Delta x_B}{\Delta x_A - \Delta x_B} |P_{\infty,12}(\Delta x_B) - P_{\infty,12}(\Delta x_A)|. \tag{13}$$

5.2. Results

The simulation is performed with different initial states and different particle momenta p_0. We set $x_0 = -1.5$ Å, $\Delta k = 10$ Å^{-1} and $\eta = 0.1$ Å. The detector parameters are $x_1 = 0$ Å, $\Delta x_1 = 0.5$ Å, $W_1 = 1 \times 10^{-3}\, mc^2$ and $x_2 = 1.26$ Å, $\Delta x_2 = 0.02$ Å, $W_2 = 1 \times 10^{-3}\, mc^2$.

Figure 3 shows the expectation values for traversal time in the detectors' rest frame K_0 with different initial states and different particle momenta p_0. The errors calculated by (12) are also plotted. The first result is that there is nearly no dependence on the initial state. In addition, the times which one obtains by using classical relativistic mechanics of a point-particle are plotted:

$$t_{t,RM} = \frac{x_2 - x_1}{c} \sqrt{1 + \frac{m^2 c^2}{p_0^2}}.$$

There is good agreement between the simulated results and those obtained by using classical relativistic mechanics. This agreement becomes more accurate with increasing particle momentum p_0.

Figure 4 shows the probability densities ρ_0 for traversal time in the detectors' rest frame K_0 with different initial states. The probability densities have a peak at the classical expected traversal time. Again, there is nearly no difference between the states $\Psi_{0,P}$ or $\Psi_{0,N}$. There are only small differences with the results obtained with the initial state $\Psi_{0,PN}$.

Next, we look at the situation in a moving reference frame K_v. It moves with velocity v relative to K_0. The traversal time in the framework of classical relativistic mechanics is

$$\tilde{t}_{t,RM} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \left[t_{t,RM} - \frac{v}{c^2}(x_2 - x_1) \right].$$

The correlation between $t_{t,RM}$ and $\tilde{t}_{t,RM}$ in classical relativistic mechanics is the same as the correlation between $T_{t,0}$ and $T_{t,v}$ in our formalism (see (8)). Again, there is good agreement

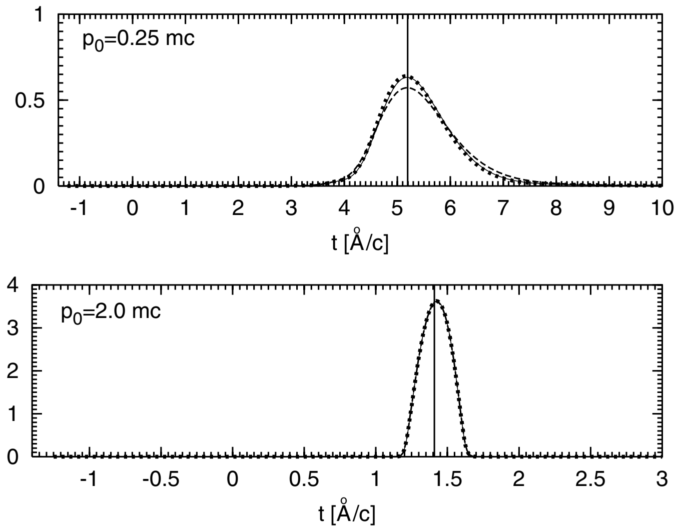

Figure 4. Probability densities ρ_0 for traversal time in the detectors' rest frame K_0, initial state: $\Psi_{0,P}$ (small solid line), $\Psi_{0,N}$ (big dotted line), $\Psi_{0,PN}$ (dashed line), particle momentum p_0; the vertical solid line indicates the traversal time given by classical relativistic mechanics.

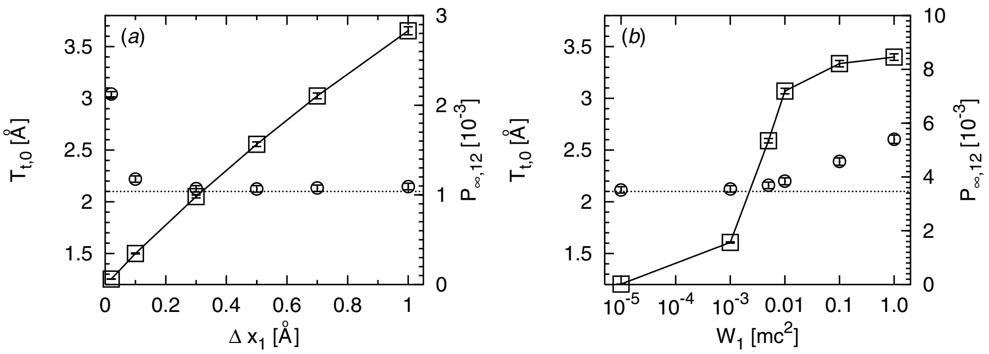

Figure 5. Mean traversal time $T_{t,0}$ (circles with error bars, left axis) and probability $P_{\infty,12}$ (boxes with error bars connected with a solid line, right axis); initial state $\Psi_{0,P}$ with $p_0 = 0.75\,mc$; detector D_2: $\Delta x_2 = 0.02$ Å, $W_1 = 1 \times 10^{-3}\,mc^2$; the dotted line indicates the traversal time deduced from classical relativistic mechanics; (*a*) detector height $W_1 = 1 \times 10^{-3}\,mc^2$; (*b*) detector width $\Delta x_1 = 0.5$ Å.

between our results and those obtained in classical relativistic mechanics in all reference frames.

Now, let us examine how the results depend on the detector parameters. The particle momentum is fixed at $p_0 = 0.75\,mc$ and the initial quantum state is $\Psi_{0,P}$. The parameters of the first detector D_1 are varied and those of the second detector D_2 are fixed at $\Delta x_2 = 0.02$ Å and $W_2 = 1 \times 10^{-3}\,mc^2$.

First, we compute $P_{\infty,12}$ and the expectation value $T_{t,0}$ in K_0 for different detector widths Δx_1 while keeping $W_1 = 1 \times 10^{-3}\,mc^2$ fixed (see figure 5(*a*)). There exists a range of detector width (0.3 Å $\lesssim \Delta x_1 \lesssim$ 1.0 Å) for which the expectation value $T_{t,0}$ does not change in a significant way. But the probability for two detections $P_{\infty,12}$ increases with increasing detector width Δx_1. In the range $0.3\,\text{Å} \lesssim \Delta x_1 \lesssim 1.0\,\text{Å}$ the forms of the probability densities ρ_0 do not differ in a significant way. The peaks only become wider with increasing detector width Δx_1. If the detector width is very small ($\Delta x_1 = 0.02\,\text{Å}$), the wavefunction changes strongly through the detection by D_1 and the probability density ρ_0 is qualitatively different.

Now let us fix $\Delta x_1 = 0.5\,\text{Å}$ and vary W_1 (see figure 5(*b*)). In the case of weakly intrusive detectors $W_1 \lesssim 5 \times 10^{-3}\,mc^2$, the expectation values $T_{t,0}$ do not differ in a significant way. For higher detectors, the expectation values $T_{t,0}$ increase a bit with increasing detector height W_1. The probability $P_{\infty,12}$ increases with increasing W_1, a fact one expects intuitively. With increasing detector height W_1, the peak of the probability densities ρ_0 is shifted to higher traversal times.

In the last part of this section, we fix the parameters of D_1 at $\Delta x_1 = 0.5\,\text{Å}$ and $W_1 = 1 \times 10^{-3}\,mc^2$ and vary the parameters Δx_2 and W_2 of detector D_2. The following pairs of detector parameters are examined: $\Delta x_2 = 0.02\,\text{Å}/W_2 = 1 \times 10^{-3}\,mc^2$, $\Delta x_2 = 0.02\,\text{Å}/W_2 = 1.0\,mc^2$, $\Delta x_2 = 0.5\,\text{Å}/W_2 = 1 \times 10^{-3}\,mc^2$ and $\Delta x_2 = 0.5\,\text{Å}/W_2 = 1.0\,mc^2$. The resulting probability densities ρ_0 and expectation values $T_{t,0}$ are nearly the same in the first three cases. The only exception is the case of a very wide and 'height' detector (last case). In that case, the mean traversal time $T_{t,0}$ is lower than in the other cases. The probability $P_{\infty,12}$ grows significantly if one increases the detector width Δx_2 or the detector height W_2. There is the same qualitative dependence of $P_{\infty,12}$ on the parameters of detector D_2 as on the parameters of detector D_1.

Note that the following fact is true in the case of weakly intrusive detectors ($W_1 = W_2 = 1 \times 10^{-3}\,mc^2$): the dependence of $T_{t,0}$ on Δx_1 is 'stronger' than the dependence on Δx_2. The reason for this is clear: changing the width Δx_1 of the first detector D_1 changes not only the first 'detection-time' but also the form of the wavefunction after the first detection.

Summarizing, there is a wide range of parameters of D_1 and D_2 for which the mean traversal time does not change significantly. Remember that the same result was found in the study of 'time of arrival'.

6. Conclusion

In this paper, the time of arrival and the traversal time of a free particle with spin $\frac{1}{2}$ has been calculated using the covariant, relativistic extension of EEQT proposed in [17].

We have found out that there is good agreement between the expectation values of our simulation and the results obtained by using classical relativistic mechanics of a free point particle. Moreover, this agreement is independent of the reference frame and holds for a wide range of detector parameters. In general, in our algorithm the particle state consists of positive and negative energy parts (after a detection this is always the case). Nevertheless, it has been shown that one can deal in these applications consistently with a one-particle interpretation.

The results presented encourage us to use the algorithm in the future to examine also moving detectors or a particle affected by a potential barrier.

Let us conclude this paper with some comments on two assumptions included in our algorithm: first, we have assumed for simplicity that the events are associated with points in spacetime, but the algorithm can be also formulated by relating the events with regions of the spacetime. The other assumption is that no event can be placed in the backward light-cone of the previous event. This condition is also chosen because it is simple and covariant. It is also possible to choose, for example, the condition in the following way: let us associate with every event a 'rest frame' (e.g. the rest frame of the detector) in which the event happens at time $t = 0$. If an event happens, we choose its rest frame and demand that the following event must happen at a time $t > 0$. The results would not change significantly except for the position of the cut (for example in figure 2 it would be at $t = 0\,\text{Å}/c$).

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